MA 635, Fall 2014

Assignment 1.

This homework is due *Thursday*, September 4.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much.

1. Quick cheat-sheet

Reminder. On the set $\mathbb{R}$ of real numbers there two binary operations, denoted by $+$ and $\cdot$ and called addition and multiplication, respectively. These operations satisfy the following properties:

(A1) $(a+b)+c = a+(b+c)$ for all $a, b, c \in \mathbb{R}$,
(A2) $a+b = b+a$ for all $a, b \in \mathbb{R}$,
(A3) there exists $0 \in \mathbb{R}$ s.t. $0+a = a+0 = a$ for all $a \in \mathbb{R}$,
(A4) for each $a \in \mathbb{R}$ there exists an element $-a$ s.t. $a+(-a) = (-a)+a = 0$,
(M1) $(ab)c = a(bc)$ for all $a, b, c \in \mathbb{R}$,
(M2) $ab = ba$ for all $a, b \in \mathbb{R}$,
(M3) there exists $1 \in \mathbb{R}$ s.t. $1 \cdot a = a \cdot 1 = a$ for all $a \in \mathbb{R}$,
(M4) for each $a \neq 0$ in $\mathbb{R}$ there exists an element $1/a$ s.t. $a \cdot (1/a) = (1/a) \cdot a = 1$,
(D) $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$ for all $a, b, c \in \mathbb{R}$.
(NT) $1 \neq 0$.

Reminder. Let $\mathbb{A}$ be a set with two operations $+$ and $\cdot$ satisfying A1–A4, M1–M3 and D, NT. (For example, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$.) The set $\mathcal{P} \subset \mathbb{A}$ is called the set of *positive elements* if

(P1) If $a, b \in \mathcal{P}$, then $a+b \in \mathbb{P}$ and $ab \in \mathbb{P}$,
(P2) If $a \in \mathbb{A}$, then exactly one of the following holds: $a \in \mathcal{P}$, $a = 0$, $-a \in \mathcal{P}$.

Then we say $a < b$ if and only if $b - a \in \mathcal{P}$; $a \leq b$ if and only if $b - a \in \mathcal{P} \cup 0$.

2. Exercises

(1) (Exercise 1.1.1 in Royden–Fitzpatrick) Let $a, b \in \mathbb{R}$. For $a \neq 0$ and $b \neq 0$, show that $(ab)^{-1} = a^{-1}b^{-1}$. (Hint: check that $a^{-1}b^{-1}$ satisfies definition of $(ab)^{-1}$.)

(2) Show that $a \cdot 0 = 0$ for all $a \in \mathbb{R}$.

(3) (1.1.2) Verify the following:
   (a) For each real number $a \neq 0$, $a^2 > 0$. In particular, $1 > 0$ since $1 \neq 0$ and $1 = 1^2$.
   (b) For each positive number a, its multiplicative inverse a^{-1} also is positive.
   (c) If $a > b$, then
   $$ac > bc \text{ if } c > 0 \quad \text{and} \quad ac < bc \text{ if } c < 0.$$
   (Hint: determine whether $ac - bc \in \mathcal{P}$.)

— see next page —

(4) In each case below, determine if P is a set of positive elements (i.e. if P satisfies P1–P2).
   (a) $\mathbb{A} = \mathbb{Z}$, $P = \mathbb{N}$,
   (b) $\mathbb{A} = \mathbb{Z}$, $P = -\mathbb{N}$,
   (c) $\mathbb{A} = \mathbb{Q}$, $P = \{r \in \mathbb{Q} : \ r > 1\}$,
   (d) $\mathbb{A} = \mathbb{C}$, $P = \{z = x + iy \in \mathbb{C} : \ x > 0\}$,
   (e) Prove that for $\mathbb{A} = \mathbb{C}$, there is no set of positive elements. (In other words, one cannot endow $\mathbb{C}$ with a meaningful order.)

(5) (1.1.4) Let a, b be real numbers.
   (a) Show that if $ab = 0$ the $a = 0$ or $b = 0$. (Hint: multiply ab by a^{-1}.)
   (b) Verify that $a^2 - b^2 = (a - b)(a + b)$ and conclude that from part (a) that if $a^2 = b^2$, then $a = b$ or $a = -b$.
   (c) Let c be a positive real number. Define $E = \{x \in \mathbb{R} \mid x^2 < c\}$. Verify that E is nonempty and bounded above. Define $x_0 = \sup E$. Show that $x_0^2 = c$. Use part (b) to how that there is a unique $x > 0$ for which $x^2 = c$. It is denoted $\sqrt{c}$.
   (d) Prove that $\sqrt{2} \notin \mathbb{Q}$.

(6) (1.1.7+) The *absolute value* $|x|$ of a real number x is defined to be $|x| = x$ if $x \geq 0$ and $|x| = -x$ if $x < 0$. For real numbers a, b verify the following:
   (a) $|ab| = |a||b|$.
   (b) (Triangle inequality) $|a + b| \leq |a| + |b|$.
   (c) (Triangle inequality) $|a - b| \geq ||a| - |b||$.
   (d) For $\varepsilon > 0$,
$$|x - a| < \varepsilon \text{ if and only if } a - \varepsilon < x < a + \varepsilon.$$

(7) (1.2.12) Problems 5c, 5d prove existence of at least one irrational number (*irrational* means "real but not rational"). Granted that at least one irrational number exists, prove that irrational numbers are dense in $\mathbb{R}$.